Comment on "Intermittency in chaotic rotations"

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Lai *et al.* [Phys. Rev. E **62**, R29 (2000)] claim that the angular velocity of the phase point moving along the chaotic trajectory in a properly chosen projection (the instantaneous frequency) is intermittent. Using the same examples, namely the Rössler and the Lorenz systems, we show the absence of intermittency in the dynamics of the instantaneous frequency. This is confirmed by demonstrating that the phase dynamics exhibits normal diffusion. We argue that the nonintermittent behavior is generic.

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Phase dynamics of chaotic oscillators attracts a large interest [1–7]. It was demonstrated that the behavior of many real-world and model autonomous chaotic systems can be regarded as oscillations with irregular amplitude and phase. The amplitude is chaotic, while the phase, as the variable corresponding to the time shifts (and thus to the zero Lyapunov exponent), exhibits a random-walk-type motion. The mean velocity of the phase growth can be interpreted as the mean frequency of the chaotic oscillator, and the diffusion of the phase is due to fluctuations of the instantaneous frequency.

Recently, Lai *et al.* [8] claimed that the dynamics of the instantaneous frequency typically exhibits the on-off intermittency. We note that speaking about "chaotic rotations," they consider not rotations in a physical space (e.g., rotations of a mechanical pendulum), but rotations of the phase point in a projection of a strange attractor, i.e., rotations here are the phase-space representation of chaotic oscillations. Lai *et al.* [8] state that the intermittency is an inherent generic property of continuous-time chaotic systems. As examples, the Rössler and the Lorenz systems are considered in [8]. In this Comment we argue that the dynamics of the instantaneous frequency is not intermittent both in the general case, and in the mentioned examples in particular.

As the presence of intermittency in [1] is mainly argued by means of the visual inspection of the time course of the instantaneous frequency, we first show that this inspection is misleading. For this purpose we consider the Rössler system with the same parameters ($a=0.165$, $b=0.2$, $c=10$) as in [1]

$$\frac{dx}{dt}=-y-z,$$

$$\frac{dy}{dt}=x+ay, \qquad (1)$$

$$\frac{dz}{dt}=b+(x-c)z.$$

If we introduce the phase as $\phi=\arctan(y/x)$, then the instantaneous frequency $\omega=\dot{\phi}$ is

$$\omega=1+\frac{axy+yz}{x^2+y^2}, \qquad (2)$$

i.e., the frequency is just an observable of the dynamical system (1), and, generically, is neither better nor worse than any other observable. To demonstrate that this observable possesses no special intermittent properties, we plot in Fig. 1

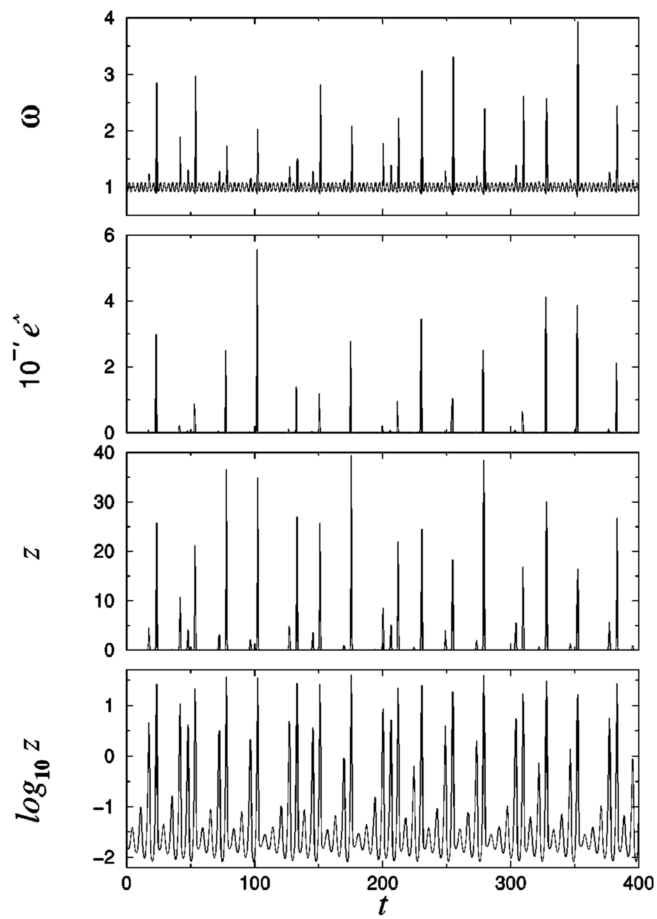

FIG. 1. The time dependencies of different observables for the Rössler system: instantaneous frequency $\omega(t)$ computed according to Eq. (2), $e^{x(t)}$, $z(t)$, and $\log_{10} z(t)$.

*URL: http://www.stat.physik.uni-potsdam.de its time dependence, together with time dependencies of some other observables. One can see that all these observables have pronounced peaks. Moreover, the peaks of $\omega(t)$ are clearly correlated with the peaks of $z(t)$. This is not surprising, because according to Eq. (2) the observable ω is proportional to z. In fact, only large peaks of z are seen in ω. To characterize the peaks, one can consider a distribution of times between them. If one, e.g., takes every maximum of z, then this distribution is quite narrow, as the return times to the Poncaré surface of section for the Rössler system are bounded ($5.4<T<6.4$ for the given parameters). Hence, the absence of the intermittency is obvious. If some cut-off level is introduced and only large maxima are taken into account, the intervals between maxima can be large, but the distribution is nevertheless exponential, as is argued below.

As for analytic arguments presented in [8], one can mention that their Eq. (1) is misleading, because the functions $\alpha(t)$ and $\beta(t)$ cannot be considered as Ω-independent. Moreover, the statement that the mean values of these variables vanish is not correct ($\langle\alpha\rangle\approx0.0825$, $\langle\beta\rangle\approx-0.16$).

As the second example we consider the Lorenz model:

$$\frac{dx}{dt}=10(y-z),$$

$$\frac{dy}{dt}=28x-y-xz, \tag{3}$$

$$\frac{dz}{dt}=-8/3z+xy.$$

The phase portrait of this system in the variables $u=\sqrt{x^2+y^2},z$ looks like rotations around a center placed near $u=12$, $z=27$. One can introduce the phase as $\phi=\arctan(\overline{z}/\overline{u})$, where $\overline{z}=z-27$, $\overline{u}=u-12$. Then the instantaneous frequency is

$$\omega=\frac{\dot{z}u\overline{u}-\overline{z}(x\dot{x}+y\dot{y})}{u(\overline{u}^2+\overline{z}^2)}. \tag{4}$$

Another possibility to compute the instantaneous frequency—by means of the Hilbert transform—has been suggested in [3,5]. One can introduce the phase, e.g., as $\phi_H=\arctan(H[\overline{u}]/[\overline{u}])$, where H denotes the Hilbert transform. (Certainly, one could shun the coordinate transformation and simply take the oscillatory observable z; we use the variable u to compare our results with those of Lai *et al.* [8].) For correct computation of the instantaneous frequency, it is convenient to avoid numerically unstable differentiation of the phase and to use the formula

$$\omega_H=\frac{\overline{u}H[\dot{u}]-\dot{u}H[\overline{u}]}{(H[\overline{u}])^2+\overline{u}^2}, \tag{5}$$

based on the fact that the differentiation and the Hilbert transform commute. We present the calculations of the frequency according to Eqs. (4) and (5) in Fig. 2. One can see that both quantities demonstrate no intermittency. We suppose that the intermittency reported in Fig. 2(d) of [8] is either an effect of numerical differentiation, or is due to not fully correct application of the Hilbert transform method. Indeed, for the correct determination of the phase, the origin on the plane $(H[\overline{u}],\ \overline{u})$ must lie in the ''hole'' of the phase portrait projection on this plane. This is ensured if one chooses $\overline{u}=u-12$ as we have done above. If, however, one takes $\overline{u}=u-\langle u\rangle$, some trajectories pass through the origin yielding spurious singularities of the instantaneous frequency.

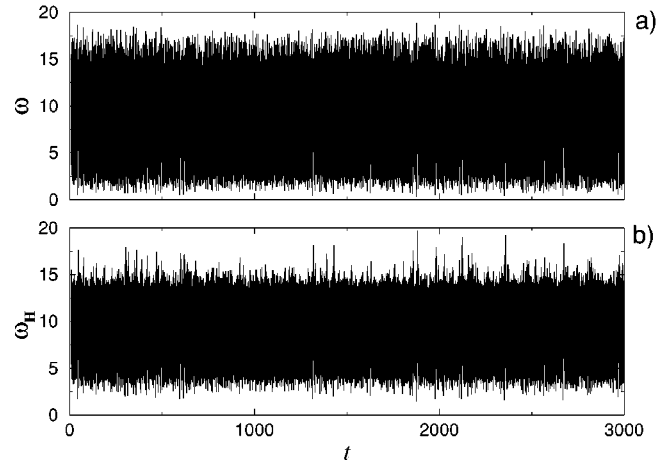

FIG. 2. (a) Instantaneous frequency $\omega(t)$ of the Lorenz system computed according to Eq. (4). (b) Instantaneous frequency $\omega_H(t)$ obtained by means of the Hilbert transform [see Eq. (5)]. Both estimates of the frequency provide consistent results (e.g., the mean frequencies coincide) and, contrary to the results of [8], exhibit no intermittency.

Another indication of the absence of the intermittency in the instantaneous frequency is obtained by characterization of phase diffusion. If the frequency were intermittent, the phase would not demonstrate the normal diffusion, but an anomalous one. In Fig. 3 we show that phase dynamics of the Lorenz system obeys the scaling of the usual diffusion $\langle(\Delta\phi)^2\rangle\propto\tau$, contrary to the claims of [9] that the phase per-

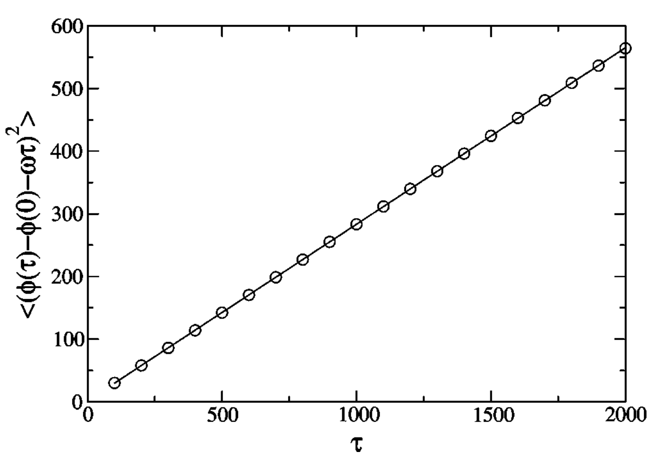

FIG. 3. The variance of the phase difference in the Lorenz system grows linearly with the time lag τ, as it should be for the normal diffusion.

forms the fractional Brownian motion. The same is true for the Rössler system: the phase diffusion there is also normal [5].

We now discuss a general question, whether the appearance of peaks in the time course of some observables can be interpreted as an indication for some kind of intermittency. Generically, the answer is negative. Indeed, let us for simplicity start with consideration of a discrete-time dissipative chaotic system, and take as an observable the characteristic function of some small subset of the attractor. It means that our observable is unity when the system visits some specified tiny region in the phase space, and zero otherwise. Clearly, the time course of such an observable will demonstrate rare peaks. Nevertheless, due to mixing, the distribution of time intervals between the peaks is exponential (very much similar to the exponential distribution of escape times in a transient chaotic state, see, e.g., [10]), i.e., the process can be approximately considered as a Poissonian one. It is a matter of taste if one calls the Poissonian point process intermittent, but we have never seen this in the literature.

One usually speaks of intermittent dynamics in the situations, where there exists a time scale (usually the duration of the laminar phase) that tends to infinity at some critical values of parameters (cf. intermittent transition to chaos according to Pomeau and Manneville [11], chaos-chaos intermittency at crisis [12], modulational (on-off) intermittency [13–15], spatiotemporal intermittency [16], eyelet intermittency [17,18]). Typically, this criticality corresponds to a ''order-chaos'' or ''chaos-chaos'' transition. Such a transition is absent in a general chaotic system under consideration.

The situation does not change for typical continuous-time dissipative systems: for these systems the return times of the Poincaré map are bounded from below and from above, and thus the exponential decay of distribution of time intervals between the peaks is preserved. This is the case for the Rössler system under consideration.

The situation is slightly more complex for the Lorenz system, where the return times are not bounded from above, due to slowing down of the motion along the trajectory near the saddle fixed point at the origin. This, however, does not lead to intermittency, because the probability to have a large return time is exponentially small (and does not follow a power law, as is claimed in [8] in the caption to Fig. 3).

Thus, one can conclude that general observables of typical chaotic systems are not intermittent. In particular, there is no ''intermittency in chaotic rotations.''

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